Short Communication

Statistical origins of fractional derivatives in viscoelasticity

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Abstract

Many linear viscoelastic materials show constitutive behavior involving fractional order derivatives. Linear, time invariant systems without memory have exponential decay in time but, contradictorily, not the power law decay associated with fractional derivatives. The physics literature has noted that apparently-non-exponential decays can be observed when several simultaneously decaying processes have closely spaced exponential decay rates. Many engineer–researchers interested in viscoelastic damping, however, seem unaware of these observations. In this letter I give an unoriginal explanation, but with a fresh engineering flavor, for the appearance of these fractional order derivatives. By this explanation, fractional order damping can be expected from many materials with sufficiently disordered dissipation mechanisms.
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1. Introduction

The aim of this letter is to interpret a puzzle from linear viscoelasticity in light of some results obtained in the physics literature [1–3]. In particular, I offer an unoriginal but more informal explanation, with a fresh engineering flavor, for the appearance of fractional order derivatives in the constitutive relations of linear viscoelastic materials.

Many linear viscoelastic damping materials exhibit a macroscopic constitutive behavior involving fractional order derivatives (see Refs. [4,5] and references therein). Such behavior has been the subject of many investigations (for a representative sample, see Refs. [6–10]).

The fractional derivative of a function $x(t)$, assuming $x(t) \equiv 0$ for $t < 0$, is (adopting the Riemann–Liouville definition used by Bagley and Torvik [4]; for a fuller treatment see Ref. [11])

$$D^{1-\alpha}[x(t)] = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} \, \mathrm{d}\tau, \quad 0 < \alpha < 1,$$

where Γ represents the gamma function. Observe that

$$\frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\tau_+^{-\alpha}}{(t-\tau)^{1-\alpha}} \, \mathrm{d}\tau = \frac{\pi}{\sin \pi\alpha \Gamma(\alpha)} \, \delta(t),$$

where $\delta(t)$ is the Dirac delta function, and where $\tau_+ = \tau$ if $\tau > 0$ and $\tau_+ = 0$ otherwise. So, if a system obeys

$$D^{1-\alpha}[x(t)] = h(t) \tag{1}$$

and has initial conditions $x(t) \equiv 0$ for $t \leqslant 0$, and if $h(t)$ is an impulse at zero, then $x(t) = Ct^{-\alpha}$ for $t > 0$ and some constant C (power law decay to zero).

A general constitutive relation used for many viscoelastic materials involves fractional order derivatives of both stress and strain, and in the one-dimensional case is

$$\sigma(t) + bD^{\alpha_1}[\sigma(t)] = E_0\varepsilon(t) + E_1 D^{\alpha_2}[\varepsilon(t)],$$

where σ is stress, ε is strain, and there are five fitted parameters on which there are some thermodynamic constraints (such as $\alpha_1 = \alpha_2$, for example; see Refs. [5,9]). Some authors (e.g., Refs. [6,7]) use the simpler

$$\sigma(t) = E_0\varepsilon(t) + E_1 D^{\alpha}[\varepsilon(t)].$$

In the above, the high-frequency behavior is dominated by the E_1 term in comparison with the E_0; and for that regime we may use as an approximation

$$\sigma(t) = E_1 D^{\alpha}[\varepsilon(t)]. \tag{2}$$

The above simplification is inessential, but clarifies the presentation. Readers unconvinced by the simplification may note that Eq. (2) has, in its own right, been studied as an "isolated fractional dashpot." By Eq. (1), the strain in a piece of material obeying Eq. (2) can have power law decay in time. That power law is the key to the appearance of fractional derivatives in the mathematical description of constitutive behavior.

2. Internal variables

In fractional order damping, the stress at a point depends explicitly on the strain history. The continuum material point has noninfinitesimal memory (unlike the infinitesimal memory needed for integer order derivatives in, e.g., Newtonian viscosity). This apparent memory must be due to the net effect of unmonitored internal processes which could, in principle, be incorporated in a larger model with internal variables but without noninfinitesimal memory. For example, consider the spring-dashpot system in Fig. 1. Point P is held fixed, R is an internal point, and force F acts on Q. L_1 and L_2 are constants such that, at equilibrium with $F \equiv 0$, $x_1 = x_2 = 0$. The two variable
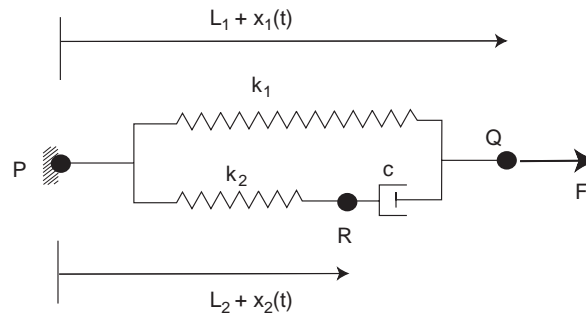
Fig. 1. A model with an internal variable.

model does not involve history explicitly. The governing equations are

$$F = k_1 x_1 + k_2 x_2, \tag{3}$$

$$c(\dot{x}_1 - \dot{x}_2) = k_2 x_2. \tag{4}$$

Now let x_2 be an unmonitored internal variable. Let us relate F and x_1. On starting from equilibrium ($x_1 = x_2 = 0$) at $t = 0$, Eq. (4) gives $x_2(t)$, which by Eq. (3) gives

$$F = (k_1 + k_2)x_1 - \frac{k_2^2}{c} \int_0^t x_1(\tau) e^{-k_2(t-\tau)/c} \, d\tau. \tag{5}$$

Thus, elimination of the hidden variable leads to a constitutive relation explicitly involving noninfinitesimal history.

3. Fractional order power laws

So far, all is straightforward. The puzzle of fractional order viscoelasticity is the source of the power law kernel (instead of an exponential one like in Eq. (5)) in the integral of history. The power law kernel is closely related to power law decay in solutions, as discussed above. Strictly speaking, fractionally damped materials involve Mittag–Leffler functions, and the associated kernels have sums of many powers instead of a single power law; however, for simplicity, we retain the single power law.

Linear constant-coefficient systems of differential equations have solution components that decay like exponentials, possibly multiplied by polynomials, but not like $t^{-\alpha}$. Where does the $t^{-\alpha}$ come from? I offer an informal explanation here for the existence of solutions like $t^{-\alpha}$ in randomly chosen systems relevant to viscoelastic damping. The mathematical solutions obtained below are not tied to the material specific physics of any particular substance. Fractional order derivatives may thus arise from microstructural mechanisms of energy dissipation which, while disordered, are less remarkable than previous discussions might suggest. As I stated earlier, my results are not new; I am simply trying to reach a different audience from that of Refs. [1–3].

Consider the model sketched in Fig. 2. An elastic rod of length L has a distributed stiffness $b(x) > 0$. Its axial displacement is $u(x, t)$. The internal force at x is $b(x) u_x$, and interaction with
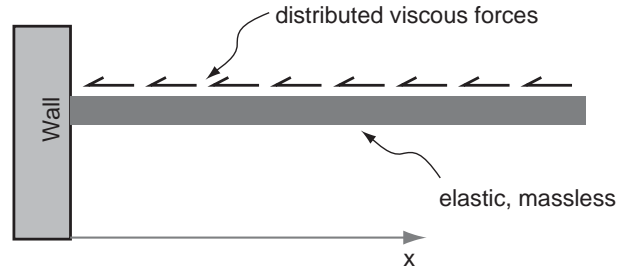
Fig. 2. One-dimensional viscoelastic model.

neighboring material causes viscous forces $c(x)u_t$, with $c(x)>0$ and with x and t subscripts denoting partial derivatives. The free end of the rod is displaced, held for some time, and released. Subsequent motion obeys

$$(b(x)u_x)_x - c(x)u_t = 0. \tag{6}$$

The boundary conditions are $u(0,t) = 0$ and $u_x(L,t) = 0$. A solution is sought in the form

$$u(x,t) = \sum_{i=1}^{n} a_i(t)\phi_i(x),$$

where large n gives accuracy, the $a_i(t)$ are to be found, and the $\phi_i(x)$ form a chosen basis and satisfy $\phi_i(0) = 0$. Discretized equations can be obtained using the method of weighted residuals [12]. Defining symmetric positive definite matrices B and C by $B_{ij} = \int_0^L b\,\phi_{i,x}\,\phi_{j,x}\,\mathrm{d}x$ and $C_{ij} = \int_0^L c\,\phi_i\,\phi_j\,\mathrm{d}x$, and writing a for the vector of coefficients $a_i(t)$, we obtain

$$C\dot{a} = -Ba.$$

On suitable choice of ϕ_i, C is the identity matrix. Then

$$\dot{a} = -Ba.$$

In the presence of sufficiently complicated microstructural behavior, the functions $c(x)$ and/or $b(x)$ might usefully be treated as random, making B random as well.

Let us study a random B. Begin with A, an $n \times n$ matrix, with n large. Let the elements of A be random, i.i.d. uniformly in $(-0.5, 0.5)$. Let $B = A^T A$. B is symmetric positive definite with probability one. We will solve

$$\dot{x} = -Bx. \tag{7}$$

Solution is done numerically using, for initial conditions, a random $n \times 1$ column matrix x_0 whose elements are i.i.d. uniformly in $(-0.5, 0.5)$. The process is repeated 30 times, with a new B and x_0 each time. The results, for $n = 400$, are shown in Fig. 3. The solutions, though they are sums of exponentials, decay on average like $t^{-1/4}$. Why?

The explanation lies in the eigenvalues of B. The spectra of random matrices comprise a subject in their own right. Here, I require a simple result that numerics can provide. Let $n = 250$. I take a random $n \times n$ matrix B as above. Let λ_k, $k = 1, 2, \ldots, n$, be its eigenvalues in increasing order. Fig. 4 shows $\sqrt{\lambda_k/n}$ plotted against k/n. Superimposed are the same quantities for $n = 400$.
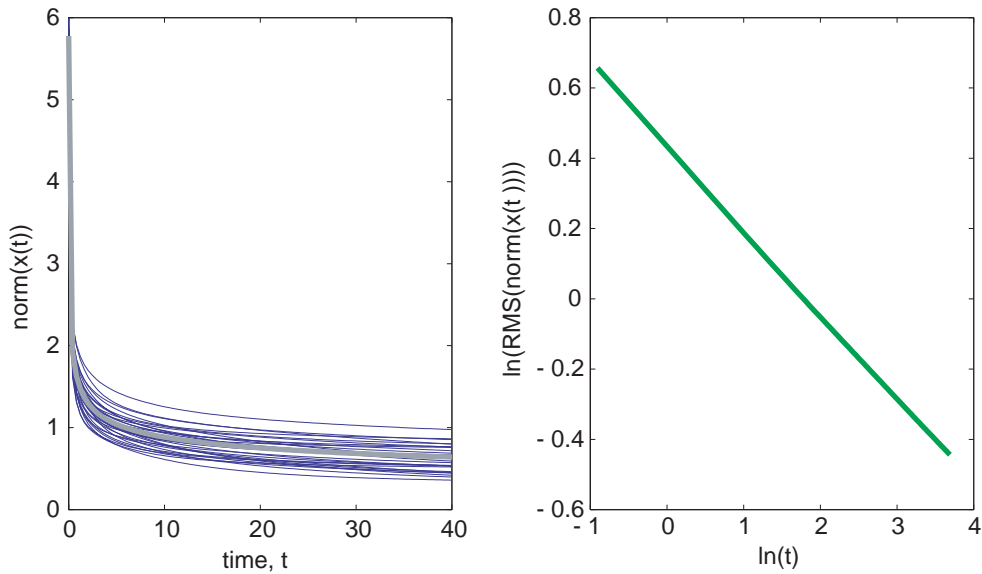
Fig. 3. Left: norm$(x) = \sqrt{x^\mathrm{T} x}$ against time. Thirty individual solutions (thin lines) as well as their rms values (thick gray). Right: rms value of norm(x) against time is a straight line on a loglog scale. A fitted line has slope $-0.24 \approx -1/4$.
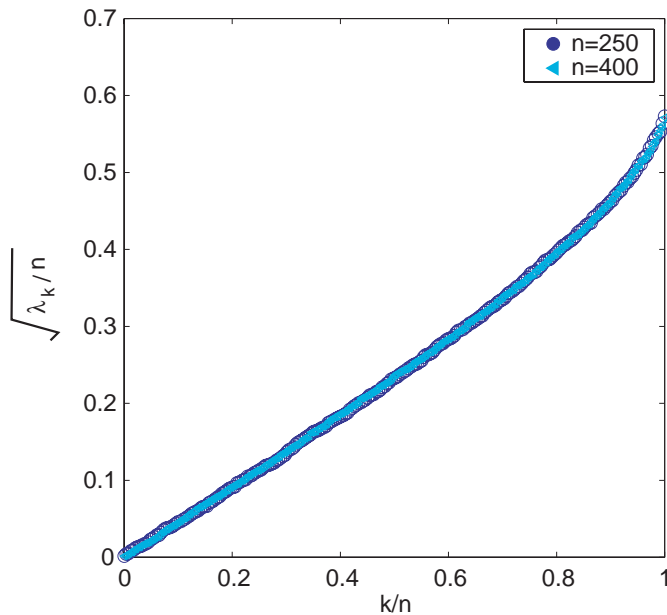

Fig. 4. Eigenvalues of B for $n = 250$ and 400.

The coincidence between plots indicates a single underlying curve as $n \to \infty$. That curve passes through the origin, and can be taken as linear if we restrict our attention to, say, the smallest 60% of the eigenvalues. Since the larger 40% correspond to rapid exponential decay, I use a linear

approximation for all eigenvalues but restrict time to values $t \gg \mathcal{O}(1/n)$, in which range solution components from the large eigenvalues have decayed to negligible values. By this approximation

$$\sqrt{\frac{\lambda_k}{n}} = \beta \frac{k}{n} \tag{8}$$

for some positive constant β. For simplicity, I ignore the statistical variation of eigenvalues around the fit of Eq. (8).

The solution for the ith element of x is of the form

$$x_i(t) = \sum_{k=1}^{n} a_{ik} e^{-\lambda_k t} = \sum_{k=1}^{n} a_{ik} e^{-\beta^2 k^2 t/n}, \tag{9}$$

where the coefficients a_{ik}, by randomness of x_0 and B and orthonormality of eigenvectors of the latter, are taken as random, i.i.d., and with zero expected value. The variance

$$\text{var}(x_i(t)) = \sum_{k=1}^{n} \text{var}(a_{ik}) e^{-2\beta^2 k^2 t/n}.$$

By scaling the initial condition x_0 suitably, we can write

$$\text{var}(a_{ik}) = \frac{1}{n\sqrt{n}},$$

independent of t and k, to obtain

$$\text{var}(x_i(t)) = \frac{1}{n\sqrt{2\beta^2 t}} \sum_{k=1}^{n} \sqrt{\frac{2\beta^2 t}{n}} e^{-2\beta^2 k^2 t/n}.$$

Define $\xi = \sqrt{2\beta^2 t/n} \, k$. Assuming $\beta^2 t \ll n$ and $n \gg 1$, the sum can be approximated by an integral. Then

$$\text{var}(x_i(t)) = \frac{1}{n\sqrt{2c^2 t}} \int_0^\infty e^{-\xi^2} \, d\xi = \frac{C^2}{n\sqrt{t}}$$

for some C. Finally, $\text{RMS}(\sqrt{x^T x})$ is (using independence of the components of x)

$$\text{RMS}(x^T x) = \sqrt{\sum_{i=1}^{n} \text{var}(x_i(t))} = \frac{C}{t^{1/4}}, \tag{10}$$

which explains the numerical result. Eventually, for t sufficiently large, the approximation fails. Only exponential decay controlled by the smallest eigenvalue remains. In experiments, the response may by then be too small to measure.

An aside on optimization. The simplest method is *steepest descents*, the continuous version of which has the form of Eq. (7). Textbooks mention the poor performance of this method for large practical problems. The power law convergence seen here gives an example of how poor that performance can be.

Acknowledgements

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